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# On generating and derived magnitudes of stellar magnetic fields

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**Abstract.** The structure of the stellar surface magnetic field is covered from direct observation by many mixing processes. The discovery of the topographic surface structure requires an inversion procedure but does not reveal the origin of the magnetic field. Modelling of magnetic stars, however, has to start from the generating magnitudes and is a matter of construction by a strategy of forward calculation. The model of the star is fitted to the observed appearance of the real object by variation of parameters and optimizing. The magnetic field strength on the surface of the star — including the magnetic poles — is a derived magnitude, which should not be taken as a parameter for modeling. At the present time two versions of magnetic modeling are discussed: 1) expansion of spherical harmonics, 2) magnetic charge distribution. Both methods claim for the application of parameters, which determine the magnetic field. In this paper the question is investigated, what the generating and the derived magnitudes of the magnetic field are. Tracing back the observed spherical distribution of the magnetic field to its origin, one is led to the eigen values as the solution of Legendre's differential equation. We regard the eigen values as the generating magnitudes of the magnetic field, the physical quantities of which are the constituents of any vector field, namely the sources and vortices, from which the field originates. This interpretation is substantiated by graphical representations of magnetic maps with topographical features like poles — derived from the field-generating sources: the virtual magnetic charges.

**Key words:** stars: chemically peculiar – stars: magnetic fields – methods: modeling

## 1 Introduction

The magnetic field of a star can be observed only in integral light radiation, which makes the recognizability of any details by many information deforming processes impossible. For the reconstruction of the original surface distribution from the final observational values all these processes have to be inverted.

The difficulties bound to the generally ill-posed inverse problem are well-known and have been considered by Khokhlova et al. (1986), who investigated at first the distribution of chemical elements over the star's surface by Doppler imaging and later, the magnetic field structure. Nevertheless, the inversion method has been developed successfully by Khokhlova herself and her followers (Piskunov 2000, Kochukhov 2003). The result of such an inversion of observational magnetic field data is a topographic distribution of the magnetic field over the surface with spotty character. The cartographic map of the field structure is still a representation of comprised observation. Not regarding the very valuable informative result, it is not clear at once, whether the often found complicated structures are the magnetic field itself or have they to be attributed to the distribution of chemical elements too. Here ends the reduction by inversion. A further reduction to the origin of the magnetic field would be very doubtful.

In contrary to the inversion, a straightforward calculation can be carried out in any case. Assuming physically reasonable conditions, models might be constructed, which are determined by parameters. A model is a simplified abstraction from the complexity of the real object. Therefore, the choice of such parameters is very important, because they are the intrinsic magnitudes of a causal process, from which any outward appearance is derived.

# 2 Modeling of stellar magnetic fields

A model of the magnetic field in a star needs at first a concept where it is coming from. The reduction of observations mark the magnetic poles as conspicuous topographical points, which determine form and time dependence of the phase curves of the integral magnetic field strength. Since we observe only the magneto-informative atmosphere, it seems to be obvious to take the magnetic field strength and the coordinates of the poles as parameters.

However, the field as a physical quantity of continuity cannot be generated in the surface layer of the star. Thus, we expect its origin either in the interior or the exterior of the star. So we ask for the intrinsic parameters of a stellar magnetic model. In any case, suitable parameters are needed, which reflect the essential characteristics of the physical conditions.

In the past, two versions of modeling stellar magnetic fields have been presented, which we will summarize briefly.

## 2.1 The Magnetic Multipolar Expansion (MME)

The modeling method based on a *multipolar expansion* of spherical harmonics gives an analytical description of a function on the sphere, which is — in case of a magnetic star — the distribution of the magnetic field strength over its surface. The coefficients of the expansion (Legendre polynomials) are varied so, that the surface distribution will be fitted to the observation. The physical meaning of the coefficients needed for the analytical formulation is not explained. From the mathematical point of view, the coefficients are the parameters of the spherical functions. Because of the complicated mathematics of Legendre functions the expansion is extended usually only up to the second degree, the quadrupole. A truncated expansion can give the main view of the star's map with the topographic sites of the poles, but it fails to describe a finer structure. The calculation of the surface field distribution by central dipoles, quadrupoles, etc. is restricted to the surface of the sphere. The magnetic surface field of a decentered or an external dipole cannot be calculated. The multipolar expansion method of modeling is an interpolating and approximation procedure for fitting and informative compressing of the observational data to an analytical representation. In this sense, it is highly developed and useful for the practical reduction of observational facts, but it does not give direct information about the origin of the magnetic field — that means, the underlying physics.

The modeling of magnetic fields in stars using Legendre's spherical functions has an old history. The first who used them for the formulation of the magnetic field structure on the surface of a star was Deutsch (1970). Further, we refer here to the papers of Oetken (1977, 1979), who modelled the star as an equatorially symmetric rotator. Oetken relates to Krause & Rädler (1980), who calculated the magnetic field structure of a star as generated by the action of a dynamo. The solution of the hydromagnetic differential equations of the dynamo is displayed as a series of Legendre functions. Of special interest are the eigen values as the solution of Legendre's differential equation, which prove to be the generating magnitudes of the magnetic field. Thus, the dynamo model is physically founded and leads immediately to an analytical description by spherical harmonics.

Spherical harmonics constitute also the mathematical basis of the modeling method of Bagnulo et al. (1996, 1998, 1999, 2001), which has been applied to a large number of magnetic stars. It comprises the statistically straying measuring values to a small set of parameters and gives a forecast of the phase curves and the line profiles for all modes of polarized light. The reduction yields field strength and coordinates of the magnetic poles on the surface, which are adopted as parameters.

### 2.2 The Magnetic Charge Distribution (MCD)

The MCD-method of modeling is founded on a theorem of the potential theory, according to which all *potentialfields* can be constructed as linear aggregates of numerous fields of point-like *sources*. The vectorial magnitude *field strength* is derived from the potential by the differential operator grad (gradient) or in the case of a vector potential by the differential operator curl (rotor). A calculus of the differential geometry states that *all* spatial vector fields can be built up by linear compilation of the fields of numerous *sources and vortices*. This holds for gravitational, hydro-dynamical, velocity, radiation, electrical, and — as well — for magnetic fields. Sources are the local points, from which the lines of force diverge. This includes also *virtual sources*, the field seems to diverge from. The sources of the field might be located anywhere in the space. The surface of a sphere — like any other plane — will be penetrated by the lines of force. Thus, decentered dipoles and external field sources produce asymmetric fields on the surface, which are calculable by a computer program with standard algorithm for the spherical field of a point-like source. The sources as the field-generating magnitudes are the solutions of Legendre's differential equation, the *eigenvalues*, which determine the field and can be used in the calculation as parameters. The sources with their fields can also combine to complex sources.

A magnetic dipole consists of two displaced magnetic sources of opposite charge. Its *magnetic moment* with the surrounding vector field is a real physical magnitude. Magnetic dipoles are the elementary bricks of any stationary magnetic field in complex combination of sources.

The MCD-method has been described by Gerth et al. (1997, 2003), Gerth & Glagolevskij (2001) and applied by (Glagolevskij & Gerth (1998), Glagolevskij et al. (1998a,b). Use of this method have also made Khalack et al. (2001a,b, 2002, 2003).

## 3 The parameters of the magnetic field

The term *parameter* is understood differently. We relate here to the mathematical sense of the parameter as a decision quantity, which distinguishes different variants of the same general concept.

If we ask for the parameters of the magnetic field, so we have at first to look for its concept. What we observe from the star, is its appearance. Likewise, the reduction of observational data by inversion calculation to the distribution of the magnetic field on the star's surface shows the appearance of the star only *better*, but does not reveal the origin of the field. A model is a hypothetical concept, which has been thought up on plausible grounds and has to be fitted to the appearance of the real object by variation of parameters and optimizing. A parameter, however, cannot be taken from the appearance of the object because it is a defining but not a derived magnitude.

Thus, also magnetic poles with their field strengths and coordinates can not serve as parameters for modeling of magnetic fields with topographic structure on the star's globe. But we do not deny that the magnetic poles with their typical surrounding field structure can define the following computation of the integral magnetic field as parameters. With an extended model of an obliquely rotating dipole, quadrupole, or multipole, the phase curve and the line profiles are *derived*.

The parameters of the magnetic field should be taken from its physical consistence where it is coming from: the *sources* and vortices, which are the generating magnitudes — suitable as parameters.

## 4 The generating magnitudes of the magnetic field

There are two ways to define the generation origin:

1) tracing back on the development path by inversion,

2) modeling by a reasonable hypothesis, and then compare the outcome with the expectation.

We start with the first way, which gives us certainty to identify the generating magnitudes. After that, we use these original magnitudes as parameters for the construction of a *Magnetic Star Model*:

### 4.1 The eigen values of the magnetic field

As is known from mathematics, a function on the sphere can be described by spherical harmonics, which is governed by the famous differential equation of Legendre:

$$(1-z^2)\frac{\mathrm{d}^2x}{\mathrm{d}z^2} - 2z\frac{\mathrm{d}x}{\mathrm{d}z} + \Big(n(n+1) - \frac{m^2}{1-z^2}\Big)x = 0. \tag{1}$$

The quantities contained in this equation are:

| | |
|---|---|
| $n$ | degree |
| $m = -n \ldots +n$ | order index, |
| $z = \cos\vartheta$ | function of azimuth angle, |
| $x = f(\vartheta,\varphi)$ | function of azimuth and longitude. |

The solutions of equation (1) are the coefficients $P_n^m(\varphi,\vartheta)$, called *"associated Legendre functions"*, which are functions of the spherical coordinates $\varphi$ (longitude) and $\vartheta$ (azimuth) at a shell with radius $r = 1$. Insofar, they describe only the spherical surface of a sphere or — in case of a star — the star's surface.

Legendre's differential equation is known in astrophysics — by the global oscillation in a star (as the sun), and in atomic physics — by the undulation atomic model (Schrödinger's equation). In both cases eigen-value solutions in the formulation by spherical harmonics play an important part. The typical combination of the integer index $n$ to $n(n+1)$ characterizes the wave or the quantum number as a discrete quantity. This is caused by the requirement of a standing undulation of the wave running on a circle around the sphere.

We do not like to give here a complete derivation of the spherical harmonics, but it might be of interest that Legendre's equation (1) follows from Laplace's equation — the homogeneous partial differential equation for a stationary potential $U$

$$\triangle U = 0 \tag{2}$$

$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ Laplace operator for Cartesian coordinates $x$, $y$, $z$

with the supposed potential function $F_n$ in spherical coordinates

$$U = r^n F_n(\vartheta,\varphi), \tag{3}$$

where $r^n$ is an exponential function of the radius with the integer exponent $n$:

$$\frac{1}{\sin\vartheta}\frac{\partial \sin\vartheta}{\partial\vartheta}\frac{\partial F_n}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial^2 F_n}{\partial\varphi^2} + n(n+1)F_n = 0\ . \tag{4}$$

We leave the complete derivation and the question *"What are the generating magnitudes of the magnetic field?"* open to the interested reader. It was our concern only to point to the fact that the solution of Legendre's differential equation can be traced back to *eigenvalues*, which take on the form of *potential sources* located in space.
But at first let us see, what the coefficients of the spherical harmonics are.

## 4.2 The magnetic field derived from spherical harmonics

With the separation concept

$$F_n(\vartheta,\varphi) = e^{im\varphi}f(\vartheta) \tag{5}$$

we obtain equation (1) in the formulation of the solution by Legendre's associated spherical polynomials. In case of a stationary potential the orthogonal vectorial components of the magnetic field $B_r$, $B_\vartheta$, and $B_\varphi$ are represented as an expansion

$$B_r = \sum_{n\geq 1}\sum_{m=-n\geq 1}^{n} A_n^m Q_n^m(\vartheta,\varphi) \tag{6}$$

$$B_\vartheta = -\frac{1}{2}\sum_{n\geq 1}\sum_{m=-n\geq 1}^{n} A_n^m \frac{\partial Q_n^m(\vartheta,\varphi)}{\partial\vartheta} \tag{7}$$

$$B_\varphi = -\frac{1}{2}\sum_{n\geq 1}\sum_{m=-n\geq 1}^{n} A_n^m \frac{\partial Q_n^m(\vartheta,\varphi)}{\partial\varphi} \tag{8}$$

$A_n^m$ constant coefficients

$Q_n^m(\vartheta,\varphi) = P_n^{|m|}(\cos\vartheta)^{\cos m\varphi}_{\sin m\varphi}$ for ${}^{m\geq 0}_{m<0}$

These formulae[1] represent a row of multipoles, whereby the (multi)pole number is the double of the degree $n$ of the polynomial. The number of the required constant coefficients $A_n^m$ is $2(2n+1)$. The row of multipoles starts by the dipole:

| | | |
|---|---|---|
| $n=1$ | dipole | 6 constants |
| $n=2$ | quadrupole | 10 constants |
| $n=3$ | sextupole | 14 constants |
| $n=4$ | octupole | 18 constants |
| . . . | | |

The multipoles may be calculated for its special degree or summed up to the highest degree required before truncation of the series. For practical purposes the expansion should be truncated, because the terms of the series grow with the degree in number and computation time of the polynomials — provided the series converges sufficiently well. The degree of the expansion, of course, determines the microstructure of the represented field distribution on the surface of the sphere. The truncation of the expansion of multipoles, however, is a violation of the physics of the magnetic star. All multipoles of the expansion are centered by definition. For a decentered dipole, the polynomials up to high degrees do not disappear and must be taken into account.

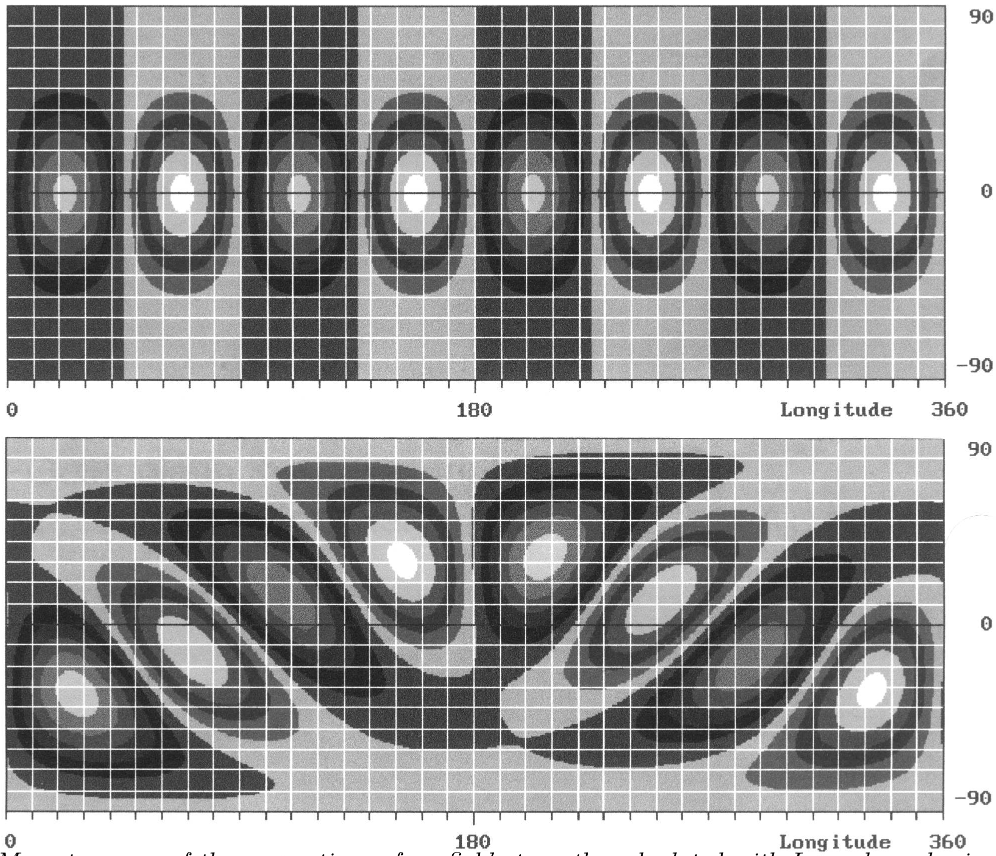


Figure 1: *Mercator map of the magnetic surface field strength, calculated with Legendre spherical harmonics (computer code written by E. Gerth).*
*Grey zones: iso-areas of equal intervals of field strength and altering polarity.*
*Top: Octupole in the equatorial belt. Only coefficient* $A_4^{-4} = 1$, *other* $A_n^m = 0$.
*Bottom: Octupole, tilted to the equator by* $40°$. *Legendre coefficients:*
$A_4^{-4} = 1$, $A_4^{-3} = 3$, $A_4^{-2} = 6$, $A_4^{-1} = 9$, $A_4^{0} = 1$, $A_4^{1} = 0$, $A_4^{2} = 0$, $A_4^{3} = 0$, $A_4^{4} = 0$.

But what is the physical meaning of the coefficients and the spherical polynomials for the magnetic field? —

[1] Equations (6–8) represent a version reduced only to the stationary potential field, which have been placed to our disposal by courtesy of Prof. K.-H. Rädler from the Astrophysical Institute Potsdam. The algorithm for the computation of the *associated Legendre polynomials* $P_n^{|m|}(\cos\vartheta)$ (written by Gerth) allows the calculation for any degree $n$ and order $m$ up to the finite accuracy of the computer. The recursion algorithm avoids the overflow of too high numbers, which occur by the faculty procedure.

If we want to construct a desired field distribution on the surface of the sphere, then we have to choose the coefficients arbitrarily — fitting them by trial and error. In principal, the coefficients are the true parameters from which the field strength on the surface is derived. This is valid, too, for the field strength on the magnetic poles, which, therefore, cannot be used as parameters for the modeling of magnetic fields on stars. So we also conclude that the coefficients of the expansion of spherical harmonics are not the generating magnitudes of the stellar magnetic field.

The program for the computation and graphical representation of the field distribution on the surface of the sphere enables one to perform numerical experiments. This was the way, the coefficients $A_n^m$ were found for the computation of Fig.1. Variation of the coefficients by trial until fitting and comparison of the given and the calculated maps is some kind of graphical correlation. A correlation algorithm is implemented also in the computer program.

We used the possibility of graphical representation of the field distribution for the investigation of the effect, the single coefficients of the spherical harmonics make on the structure of the map. We found that the map for any coefficient can be produced also by a magnetic dipole located inside the sphere. Thus, the spherical coefficients are identified as *magnetic moments*, which can be arranged in an expansion like spherical harmonics. Such a set of magnetic dipoles, however, does not reflect the real physics and cannot improve our knowledge about origin and generation of the magnetic field.

Spherical harmonics are an excellent mathematical calculus for the analytical description of functions on the surface of the sphere. Its awkward complexity, however, makes comprehension, requirement, and practical application difficult, so that we look for some appropriate simplifications.

Let us, therefore, go over from Legendre's spherical functions to the better-known case of trigonometrical functions. Then the two-dimensional distribution on the surface of the sphere with coordinates $\vartheta$, $\varphi$ is reduced to an one-dimensional oscillation as a process in time $t$.

## 4.3 The oscillation equation compared to Legendre's equation

If we specialize Legendre's differential equation (1) for a constant azimuth angle $x = \cos\vartheta = \text{const}$, we get the oscillation equation for an oscillating ring with eigen frequencies of the overtone row.

We go a step further and derive from Legendre's equation (1) immediately the differential equation for a single mechanical oscillator, replacing all in this case constant magnitudes by the appropriate mechanical ones

| | |
|---|---|
| $M \Longrightarrow 1 - z^2$ | mass, |
| $R \Longrightarrow z$ | friction resistance, |
| $D \Longrightarrow n(n+1) - \frac{m^2}{1-x^2}$ | direction force , |

then we have:

$$M\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 2R\frac{\mathrm{d}x}{\mathrm{d}t} + Dx = 0. \tag{9}$$

With the functional concept for a solution

$$x = ae^{\lambda t} \tag{10}$$

we get two possible *eigen*-solutions:

$$\lambda_1 = -\frac{R}{M} + \sqrt{\frac{D}{M} - \frac{R^2}{M^2}} \qquad \lambda_2 = -\frac{R}{M} - \sqrt{\frac{D}{M} - \frac{R^2}{M^2}}. \tag{11}$$

The general integral of the differential equation is the sum of all solutions, which is in the present case a damped oscillation, represented as a trigonometric row with two terms of amplitudes $a$ and $b$

$$x = e^{-\frac{R}{M}t}(ae^{i\omega t} + be^{-i\omega t}). \tag{12}$$

The solution is determined by the eigen frequency:

$$\omega = \sqrt{\frac{D}{M} - \frac{R^2}{M^2}}. \tag{13}$$

The generating magnitude of the oscillator, which determines the eigen frequency $\omega$, is obviously $D/M$. The damping term $R/M$ varies slightly the eigen frequency.

Numerous oscillators — as we have in a musical instrument like a piano — superpose their oscillations. A triad, for instance, consists of three eigen frequencies. If the sound of the triad is analyzed by Fourier analysis, then it would be traced back to the three eigen frequencies and its generating magnitudes — mass, length, and tension of the string.

But also the reversal is possible: if we know the generating magnitudes (as an instrument builder does), then we have the eigen frequencies already in advance and can construct and overlay all single oscillations to a sound like the triad. This is the synthetic way — as we do by constructing magnetic fields out of their potential sources.

The analogy between spherical harmonics and trigonometric functions is surprisingly close and can even be used for many practical applications. We list here some common properties:

1) formulation as differential equations;

2) reduction to eigen values;

3) solution as expansion of functional terms (functions of eigen values — eigen functions);

4) linearity and orthogonality;

5) linear superposition;

6) transformation to the complex projection space (Laplace-Transformation);

7) inverse analysis procedures
a) Fourier analysis,
b) "Legendre" analysis.

# 5 The elementary field configuration of sources and vortices

The MCD-method uses the intrinsic eigen values as the original generating magnitudes positioned in space, from which the magnetic field is derived. Usually, a complex field configuration is a derivation of a combination of eigen values, which have as solutions of a differential equation the property of linear aggregates.

Since we know that a complex field is a linear superposition of numerous fields, we can reduce the field to its elementary constituents. Therefore, we investigate a single eigen value as a generator for an elementary field in space. We can expect that such an elementary field and its analytical description has the utmost simple form suitable for generalization and programming on a computer.

The straightforward calculation is the synthetic way to build any field configuration out of the generating magnitudes.

## 5.1 Derivation of the magnetic monopole field from its source

The origin point of the field is located in spherical coordinates $\varphi$ — longitude, $\delta$ — latitude, $r$ — radius-fraction (the distance of the point from the center of the sphere in fractions of its radius $R$). The three orthogonal components $\boldsymbol{B}_r$, $\boldsymbol{B}_\varphi$, $\boldsymbol{B}_\delta$ of the field vector in the center of the surface element $\triangle$S are given by equations (18–20).

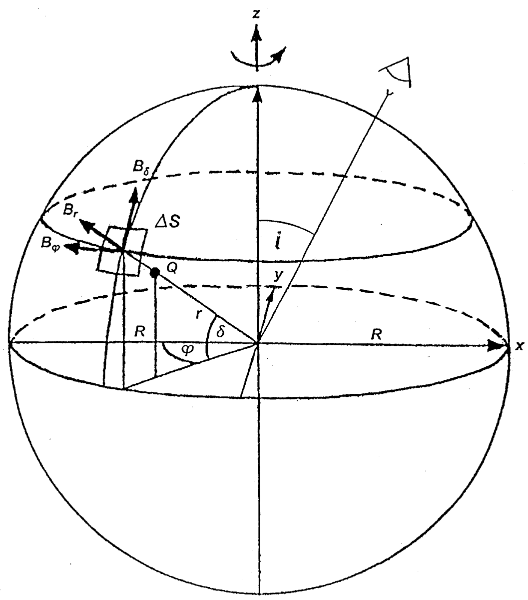


Figure 2: *Geometry of a point-like source in a sphere. The star is orientated in the Cartesian coordinate system with its rotation axis coinciding with the z-coordinate, the observer looks at by the inclination angle i.*

In Cartesian coordinates $x$, $y$, $z$ we have with the unity vectors **i**, **j**, **k** the gradient $U$ (Gerth & Glagolevskij 2001, Gerth et al. 1999). From the scalar potential $U$ the field strength is derived by the linear differential operator gradient

$$\boldsymbol{B} = -\mathrm{grad}U. \tag{14}$$

The gradient is a vector of 3 components, which span a space with 3 orthogonal unity vectors as Cartesian or spherical coordinates

$$\mathrm{grad}U = \frac{\partial U}{\partial x}\mathbf{i} + \frac{\partial U}{\partial y}\mathbf{j} + \frac{\partial U}{\partial z}\mathbf{k}. \tag{15}$$

Likewise, we have for each point of the sphere in the polar orthogonal system of radius $r$, longitude $\varphi$, and latitude $\delta$ the gradient

$$\mathrm{grad}U = \frac{\partial U}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}x}\mathbf{i} + \frac{\partial U}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}y}\mathbf{j} + \frac{\partial U}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}z}\mathbf{k}. \tag{16}$$

If we consider only the one-dimensional case using polar coordinates with radius $r$ and simplify the constant with $Q$ to $C = -\frac{Q}{4\pi}$, then the potential

$$U = -\frac{C}{r} \qquad \text{yields the gradient} \qquad \frac{\mathrm{d}U}{\mathrm{d}r} = \frac{C}{r^2}. \tag{17}$$

The magnetic monopole charge is located anywhere inside (or outside) the star and produces a magnetic field as shown in Fig. 3.

The differential quotients that give the gradient along the 3 orthogonal polar coordinates are:

$$\boldsymbol{B}_r = \partial U/\partial r = (C/r^3)[\cos\delta(\cos\varphi + \sin\varphi) + \sin\delta], \tag{18}$$

$$\boldsymbol{B}_\varphi = \partial U/\partial \varphi = (aC/r^3)\cos\delta(\cos\varphi - \sin\varphi), \tag{19}$$

$$\boldsymbol{B}_\delta = \partial U/\partial \delta = (aC/r^3)[\cos\delta - \sin\delta(\sin\varphi + \cos\varphi)]. \tag{20}$$

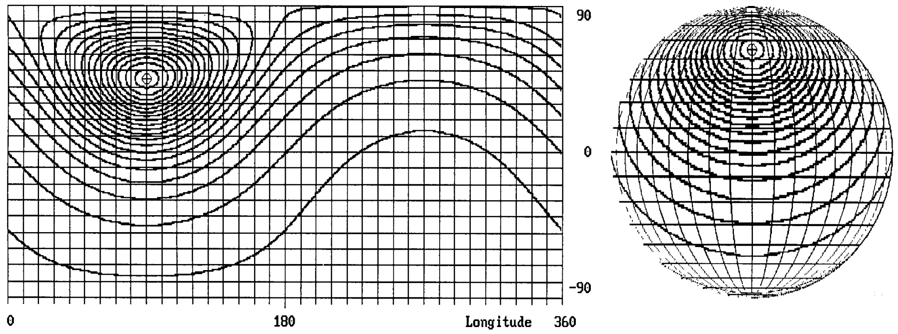


Figure 3: *Map and globe of the field structure of an eccentric monopole on the surface of a sphere. The field of a monopole is determined by four parameters (three local coordinates $x, y, z$ and charge $Q$). A monopole of unit charge is located at fractional radius $r = 0.5$, longitude $\varphi = 90°$, and latitude $\delta = 45°$.*

These equations are the basic relations for the calculation of the magnetic field strength distribution over the star's surface for a single monopole. The differential quotients represent the 3 coordinates of the magnetic field at the surface of the star, which constitute the field vector. The mapping of the magnetic surface structure relates to these values.

## 5.2 Construction of a magnetic vortex field

Like the gradient for the magnetic dipole, the calculation of the field strength for the magnetic vortex is based on the linear differential operator **curl**.

A vortex (Gerth et al. 2003) constitutes the closed magnetic lines of force around an axial vector with origin at spherical coordinates $r$, $\varphi$, $\delta$ and direction determined by the spatial motion of an electrical charge through Cartesian space. The three vector components of the electrical current $\boldsymbol{I}$, with origin at Cartesian coordinates $x$, $y$, $z$ on the sphere with radius $r$, can be written in spherical coordinates also with three parameters: the magnitude of the current $\boldsymbol{I}$, and $\lambda$, the horizontal component and $\vartheta$, the azimuthal component.
The field strength of a vortex is derived by the vectorial differential operator curl:

$$\mathrm{curl} I = \left(\frac{\partial I_z}{\partial y} - \frac{\partial I_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial I_x}{\partial z} - \frac{\partial I_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial I_y}{\partial x} - \frac{\partial I_x}{\partial y}\right)\mathbf{k}, \tag{21}$$

**i, j, k** Cartesian unit vectors; $U$ potential, $\boldsymbol{I}$ electrical current with components $I_x$, $I_y$, $I_z$.

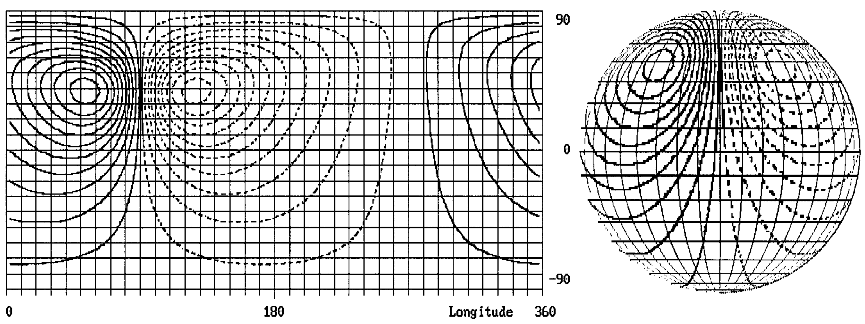


Figure 4: *Map and globe of the field of an eccentric vortex on the surface of a sphere. Solid lines: positive region; dotted lines: negative region. The field of a vortex is determined by six parameters (3 local, 3 electric). The fractional radius is $r = 0.5$, the longitude $\varphi = 90°$, the latitude is $\delta = 45°$ and $I_x = I_z = 0$, $I_y = 1$.*

The partial differential quotients of the Cartesian components of the current, $I_x$, $I_y$, $I_z$, are calculated in the same manner as the differential quotients of the potential $U$ corresponding to equation (5) with terms like equations (18–20).

We do not pursue further the construction of a magnetic field by vortices, because the stationary field is built up only by the gradient of the potential. Here we state this possibility for completeness. Evaluating Maxwell's equations as for the dynamo theory (Krause & Rädler 1980, Rädler 1995), the solution of the transformed differential equation of continuity leads to *sources* and *vortices* as *eigenvalues*.

In some cases, the computation of fields with closed lines of force might be convenient, for instance, modeling the magnetic loops in the solar atmosphere and the corona.

In any case, we have to add the vortices with their parameters to the generating magnitudes, from which a magnetic field is derived.

# 6 Superposition of magnetic fields of numerous sources

The possibility of linear superposition of magnetic fields can be taken as lucky coincidence. All complex field configurations are composed of elementary field generators requiring for the computation the same standard algorithms, which are run repeatedly for all positions in space.

## 6.1 Virtual sources

In our study we assume magnetic field sources to be point-like. The magnetic lines of force around a moving electrical charge — an electric current — are closed, after the famous law of *Biot* and *Savart*. Magnetic charges concentrated in points, however, seem to violate physics — an objection, raised to the authors frequently.

We mentioned already, that the assumption of point-like sources is a requirement of the computer program used for the calculation. But are *magnetic charges* as sources for the magnetic field wrong at all?

The magnetic field, of course, obeys the physical laws of all fields with common properties, so that we state: *All fields originate from sources and vortices.*

If we trace back the lines of force in any volume element of space by tangential elongation to their crossing point, then we hit a *virtual* source, where the lines seem to come from. This is in analogy to the virtual light sources in optics, which we see behind a mirror or a lens. In case of a monopole field the situation is clear: all lines of force are directed on straight radial lines to the center of a sphere, so that virtual and real sources coincide. The lines of force converge to the center and diverge from the center. Mathematically, this geometrical constellation is described by the differential operator *divergence*, which gives the balance of incoming and outgoing lines of force through the closed surface of a volume element. If a source is contained within the volume element, then the divergence does not disappear,

$$\mathrm{div} B \neq 0. \tag{22}$$

For a sphere, the convergence of the lines of force to the center is quite clear. However, equation (22) holds also for every closed surface around the source, also even for a source decentered in the sphere. Moreover, equation (22) holds for many sources enclosed in the volume — as a consequence of the superposition theorem of solutions of differential equations.

So we can take the expressions *divergence* and its counterpart *convergence* word for word: the lines of force seem to diverge from sites, whose real existence is not known in advance, so that we can call them generally *virtual sources*.

## 6.2 The magnetic dipole

Combinations of magnetic sources — so as the magnetic moment of a magnetic dipole — are also generating magnitudes, from which a complex field is derived. In analogy to an electric dipole, we construct a magnetic dipole by two magnetic charges $Q_1 = Q$ and $Q_2 = -Q$ of opposite polarity in a distance $\boldsymbol{l}$ from each other. The product

$$\boldsymbol{M} = Q\boldsymbol{l} \tag{23}$$

is an axial vector with a surrounding characteristic magnetic vector field, the *magnetic moment*. We introduce here the common case of a magnetic moment with a distance $\boldsymbol{l} > 0$. The infinitesimal case $\boldsymbol{l} \to 0$ does not change the value of the magnetic moment, the magnetic charges $Q_1$ and $Q_2$, however, would grow to infinity. This is the *mathematical dipole*, the field of which is assessed as the *normal dipole field.*

The magnetic dipole is in any case a real physical quantity. So, also a rigid compound of two oppositely charged sources, like a rod magnet with a north pole and a south pole, is a magnetic dipole. A steel magnet is composed of micro-magnets with atomic dimensions. The atomic magnetic moment (Bohr's *magneton*) produces a dipole field by the orbital movement of the electric charge of the electron around the nucleus. In macro dimensions also an electric current, circulating in a loop, makes a dipole field with a magnetic moment. The difference to the two-sources-dipole is only the inner structure, where all field lines penetrate the plane spanned by the loop without crossing each other. The narrower the loop, or the closer the two magnetic charges, the more both variants of dipole fields coincide with growing distance.

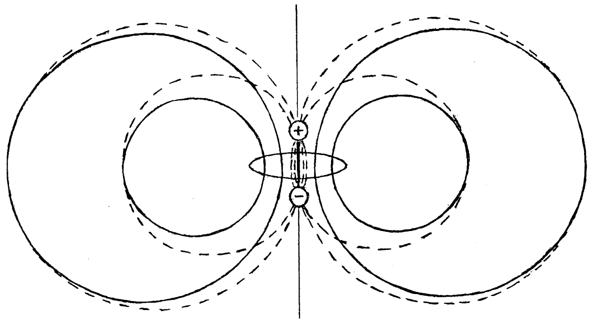


Figure 5: *Demonstration of a dipole with virtual sources. An electric current flowing through a loop produces a magnetic field configuration with closed lines of force (solid lines), which penetrate the plane of the loop without crossing. The diverging lines of force (dotted lines) can be traced back to their origin as if it was a dipole with virtual sources — in analogy to virtual images in optics.*

Fig. 5 demonstrates schematically the structure of the lines of force for a dipole of a circulating current compared with the corresponding dipole consisting of a couple of magnetic charges. The distance of the charges and the diameter of the loop have the same dimensions, within which the structures of the current-based and the charge-based dipoles differ most conspicuously. For a replacement, this region should be excluded. The lines of force, coming from the outer region, are directed on regular circles to its crossing point where they are focused in the sites of the virtual sources in analogy to optical image projection. The focussing to the virtual source might not be sharp and can have a distribution like a caustic in optics. Then we take the point of maximal concentration of field lines as the best approximation of the magnetic dipole modelled by magnetic charges. The difference between the two modes of magnetic dipoles disappears for dimensions of the electrical circuit or the distance of the charges small to the radius of the field strength plane. This is obvious for atomic dimensions of elementary magnets viewed in macro-cosmos.

In Fig. 6 the cartographic map and globes in four phases are shown of a central magnetic dipole derived from two separated magnetic charges of opposite polarity. The sources are arranged symmetrical to the center as a *central dipole*. Only for dipoles with axes through the center the coordinates of the magnetic poles on the surface $\varphi, \delta$ agree with those of the sources, which is not fulfilled for anyhow transversely decentered dipoles. The calculation of the magnetic surface field by spherical harmonics is confined only to central dipoles. Generally, the magnetic field strength — including the poles — is *derived* from the generating magnitudes: the magnetic sources.

The magnetic dipole in any form can be regarded as an elementary unit defined by the vectorial magnetic moment with its magnetic field. Thus, also the *magnetic dipole moment* is a *generating magnitude* itself, which is the *elementary brick* to build any magnetic body by composing the magnetic moments.

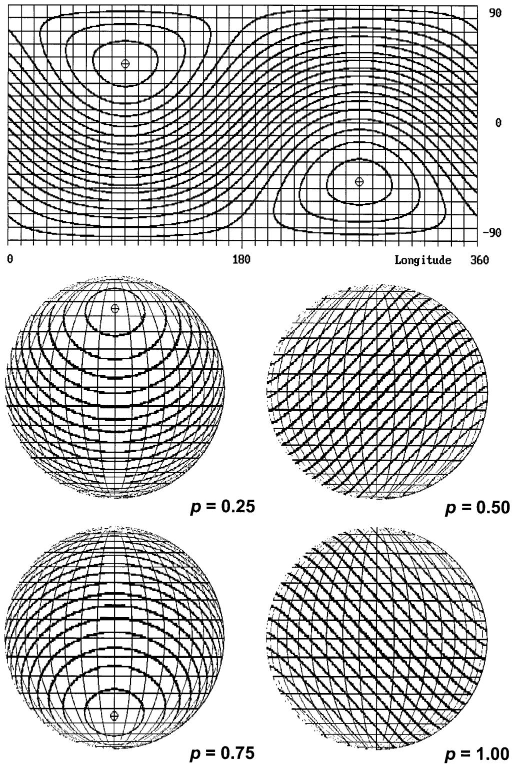


Figure 6: *Mercator map with globes to the phases 0.25, 0.5, 0.75, and 1.00 of the magnetic field with the surface elements arranged as a matrix. Parameters*:

| Charge | Longitude | Latitude | Radius-fraction |
|---|---|---|---|
| $Q_1 = +1$ | $\varphi_1 = 90^o$ | $\delta_1 = +45^o$ | $r_1 = 0.1$ |
| $Q_2 = -1$ | $\varphi_2 = 270^o$ | $\delta_2 = -45^o$ | $r_2 = 0.1$ |

*The magnetic charge $Q$ and the radius $r$ are given in relative units.*

## 6.3 Magnetic multipoles

Magnetic sources can formally be distributed in space arbitrarily. However, to preserve the connection to physics, at least the condition

$$\sum^{i} = 0 \tag{24}$$

has to be respected. Moreover, the coordination of pairs with opposite but absolutely equal charges like pairs should be kept to. So we can construct all multipoles by spatial arrangements of dipoles.

The magnetic dipole moment is an axial vector and obeys all rules of vector algebra. Combination of dipoles to multipoles is vector addition of the magnetic moments. A quadrupole can be combined by two dipoles. Two central dipoles yield again a central dipole with a magnetic moment following from the resultant of a vector parallelogram. The resultant poles lie between the poles of the summand moments. The effect of addition of magnetic moments on the map is calculated and demonstrated graphically by Gerth & Glagolevskij (2002).

Nevertheless, the topographic structure of resultant dipole field gives no information about the vector summands, because the poles are areas of the surface field, and therefore, they are *derived* from the *generating* magnetic moments.

## 6.4 Super-multipoles

The superposition of magnetic fields is a possibility to sum up numerous fields using for the calculation repeatedly the same standard algorithms.

Correspondingly to the *magnetic charges* the *elementary magnetic dipoles* may be arranged arbitrarily within the stellar body by position and by direction. The combination of elementary dipoles enables one to model different magnetic bodies: rod, cubic, cylinder, ellipsoid etc. In principle, this is valid also for macro-magnets and elementary (atomic) micro-magnets. The density of elementary dipoles determines, of course, the required computation time. The linear superposition allows also to divide the ensemble into subgroups. The possibilities are infinite. We will present here only an example for a field structure, which deviates from the normal dipole field. Especially interesting is the field of an area of a circle, set with elementary dipoles and forming a "magnetic sheet".

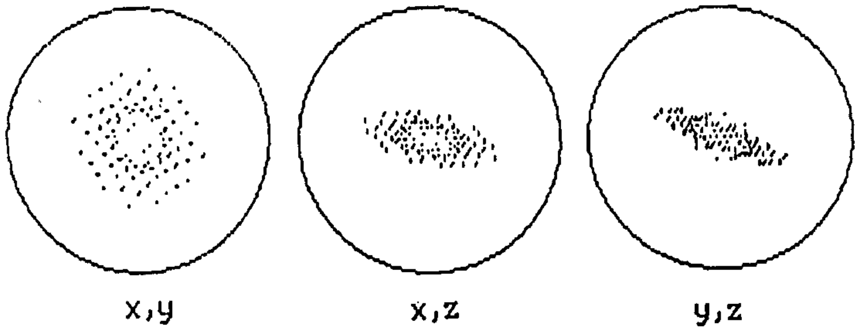


Figure 7: *"Super-multipole" of 80 dipoles as double layer of positive and negative monopoles. The grating looks pillow-like and empty in the middle because of the shifting of the charges. All dipoles have equally two oppositely charged field sources. Each dipole has a distance of the 2 point-like charges of 0.01 R (160 points). The dipoles are set within the circle r = 0.5 R in a grating of* $10 \times 10$.

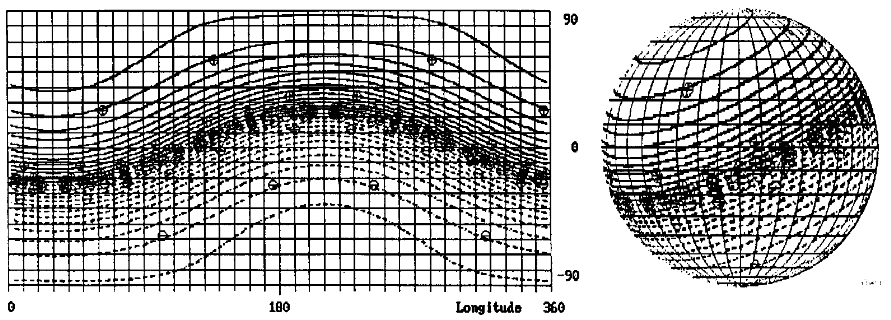


Figure 8: *Mercator-map and globe of the circular magnetic sheet approximated by 80 dipoles, represented with the cartographic coordinates of the sources* $\oplus$ $\ominus$ *and iso-magnetic lines. The sheet lies as a circular disk in the x, y-plane of the star with half of the stellar radius in the center — tilted by 30° to the x-axis and to the y-axis.*

With an arrangement of dipoles as in Fig.7 we can construct a "super-multipole" as an entirety of predetermined form, using elementary magnetic dipoles like "bricks" for a building. The resulting field structure as shown in Fig.8 is *generated* by and *derived* from the magnetic sheet. Such a magnetic field as that of a sheet is produced by a circularly streaming electric current, as we can assume circling in the star both in cases of a stellar dynamo and of a frozen-in relict magnetism.

We come back to the philosophy of the MCD-method relating to a definite theorem of the potential theory, according to which *any field configuration is produced by superposition of the fields of numerous point-like sources.*

## 7 Conclusion

The discussion on two current versions of modeling stellar magnetic fields concerns the physical foundation and the origin for assessment and practical use. The requirement of reasonable parameters for calculation rises the rather philosophical question: What was first — the magnet or the magnetic field? We tried to investigate the causal connection between generating and derived magnitudes by comparison of the two methods of *magnetic modeling.* Therefore, we outlined their physical and mathematical foundation briefly and looked for essential common and distinguishing characteristics.

Despite both methods represent different aspects of the item, they have an intrinsic logical connection without any contradiction. The link between the methods are the *eigenvalues*, which are seen from one side as solutions of Legendre's differential equation, and from the other side as the *generating magnitudes.* The decision, which method is to be applied, depends on the purpose. There are good prospects to elaborate a common theory and to bring both methods together. In any case, the development has not come to an end yet, and the possibilities of application are still not exhausted.

## References


Bagnulo S., Landi Degl'Innocenti M., Landi Degl'Innocenti E., 1996, Astron. Astroph., **308**, 115

Bagnulo S., 1998, European Working Group on CP Stars, Vienna 27–29 Oct.1997, eds.: P. North, A. Schnell, J. Žižňovsky, Contrib. Astron. Obs. Skalnaté Pleso, **27**, 431

Bagnulo S., Landolfi M., Landi Degl'Innocenti M., 1999, Astron. Astrophys., **343**, 865

Bagnulo S., Wade G.A., Donati J.-F., Landstreet J.D., Leone F., Monin D.N., Stift M.J., 2001, Astron. Astroph., **369**, 889

Deutsch A.J., 1970, Astrophys. J., **159**, 895

Gerth E., Glagolevskij Yu. V., Scholz G., 1997, Proc. Intern. Conf. on Stellar Magnetic Fields, SAO Nizhnij Arkhyz 13–18 May 1996, eds.: Yu.V. Glagolevskij, I.I. Romanyuk, 67

Gerth E., Glagolevskij Yu. V., Scholz G., 1998, European Working Group on CP Stars, Vienna 27–29 Oct. 1997, eds.: P. North, A. Schnell, J. Žižňovsky, Contrib. Astron. Obs. Skalnaté Pleso, **27**, 455

Gerth E., Glagolevskij Yu.V., Hildebrandt G., Lehmann H., Scholz G., 1999, Astron. Astroph., **351**, 133

Gerth E., Glagolevskij Yu.V., 2000, Magnetic fields of chemically peculiar and related stars, Proc. Intern. Conf., Nizhnij Arkhyz, eds.: Yu. V. Glagolev-skij, I.I. Romanjuk, Nizhnij Arkhyz 24–27 Sep.1999, 151

Gerth E., Glagolevskij Yu.V., 2001, Magnetic fields across the Hertzsprung-Russell diagram, Astr. Soc. Pac. Conf. Ser., eds.: Mathys G., Solanki S.K., Wickramasinghe D.T., Santiago de Chile 15–19 Jan.2001, **248**, 333

Gerth E., Glagolevskij Yu.V., 2002, Modelling of Stellar Atmospheres, Proc. IAU Symposium No. 210, Uppsala 17–21 Jun 2002, eds.: N.E. Piskunov, W.W. Weiss, D.F. Gray, (in press)

Gerth E., Scholz G., Glagolevskij Yu.V., 2003, Intern. Conf. On Magnetic fields in O, B and A stars, ASP Ser. 216, eds.: Balona L.A., Henrichs H, Medupe T., Mmabatho 27 Nov.–1 Dec. 2002, 373

Glagolevskij Yu.V., Gerth E, 1998, Bull. Spec. Astrophys. Obs., **46**, 123

Glagolevskij Yu.V., Gerth E., Hildebrandt G., Scholz G., 1998a, European Working Group on CP Stars, eds.: P. North, A. Schnell, J. Žižňovsky, Vienna 27–29 Oct.1997, Contrib. Astron. Obs. Skalnaté Pleso, **27**, 461

Glagolevskij Yu.V., Gerth E., Hildebrandt G., Lehmann H., Scholz G., 1998b, European Working Group on CP Stars, eds.: P. North, A. Schnell, J. Žižňovsky, Vienna 27–29 Oct.1997, Contrib. Astron. Obs. Skalnaté Pleso, **27**, 458

Khalack V.R., Khalack Yu.N., Shavrina A.V., Khalack J.N., Polosuchina N.S., 2001a, Magnetic Fields across the Hertzsprung-Russell diagram, Astr. Soc. Pac. Ser., eds.: G. Mathys, S.K. Solanki, D.T. Wickramasinghe, Santiago de Chile 15–19 Jan. 2001, **248**, 329

Khalack V.R., Khalack Yu.N., Shavrina A. V. et al., 2001b, Astron. Zh., **78**, N7, 655

Khalack V.R., 2002, Asron. Astroph., **385**, 896

Khalack, V.R. Zverko, J., Žižňovsky, J. 2003, Astron. Astroph., **403**, 179

Khokhlova V.L., Rice J.B., Wehlau W.H., 1986, Astrophys. J., **307**, 768

Kochukhov O., 2003, Thesis: Magnetic and Chemical Structures in Stellar Atmospheres, Uppsala

Krause F., Rädler K.-H., 1980, Mean-Field Magnetohydrodynamics and Dynamo Theory, Akademie-Verl., Berlin, and Pergamon Press, Oxford

Rädler K.-H., 1995, Reviews in Modern Astronomy, Astronomische Gesellschaft, ed.: G. Klare, 8, 295

Oetken L., 1977, Astron. Nachr., **298**, 197

Oetken L., 1979, Astron. Nachr., **300**, 1

Piskunov N.E., 2000, Magnetic fields of chemically peculiar and related stars, Proc. Intern. Conf., Nizhnij Arkhyz, eds.: Yu. V. Glagolevskij, I.I. Romanjuk, Nizhnij Arkhyz 24–27 Sep. 1999, 96